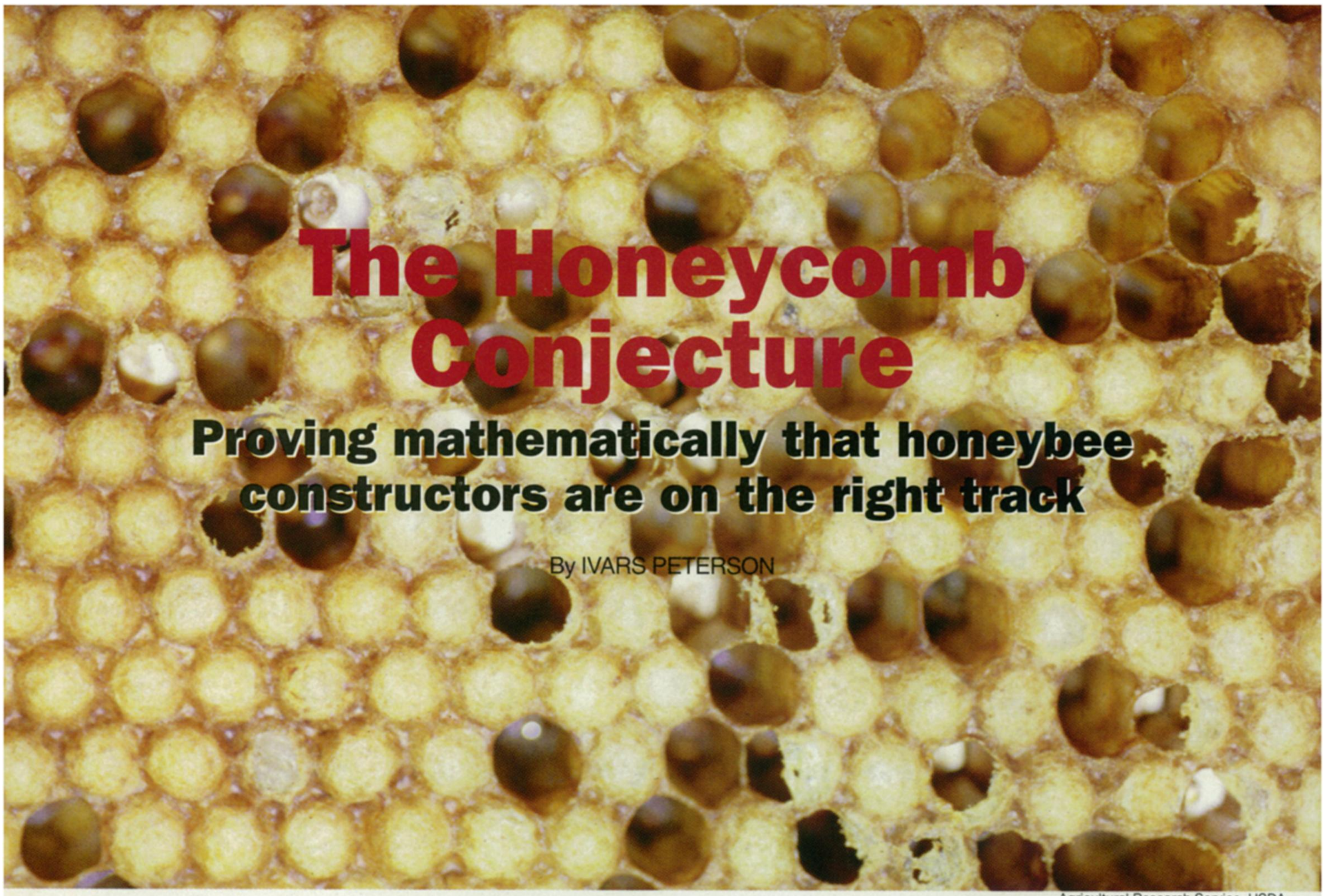
The Honeycomb Conjecture

Proving mathematically that honeybee constructors are on the right track

By IVARS PETERSON



The honeybee's storage system consists of an array of hexagonal cells precisely constructed from wax. Agricultural Research Service, USDA

Honeybees know a thing or two about working with wax and fashioning elegant, symmetrical structures.

Gorging themselves on honey, young worker bees slowly excrete slivers of wax, each fleck about the size of a pinhead. Other workers harvest these tiny wax scales, then carefully position and mold them to assemble a vertical comb of six-sided, or hexagonal, cells. The bees cluster in large numbers, maintaining a hive temperature of 35°C, which keeps the wax firm but malleable during cell construction.

This energetic, piecemeal activity produces a strong, remarkably precise structure. Each wax partition, less than 0.1 millimeter thick, is fashioned to a tolerance of 0.002 mm. Moreover, the cell walls all stand at the correct 120° angle with respect to one another to form a lattice of regular hexagons.

Observers throughout recorded history have marveled at the hexagonal pattern of the honeybee's elaborate storage system. More than 2,000 years ago, Greek scholars commented on how bees apparently possess "a certain geometrical forethought" in achieving just the right type of enclosure to hold honey efficiently. In the 19th century, Charles Darwin described the honeycomb as a masterpiece of engineering that is "absolutely perfect in economising labour and wax."

Biologists assume that bees minimize the amount of wax they use to build their combs. But is a grid made up of regular hexagons indeed the best possible choice? What if the walls were curved rather than flat, for example?

Mathematician Thomas C. Hales of the University of Michigan at Ann Arbor has now formulated a proof of the so-called honeycomb conjecture, which holds that a hexagonal grid represents the best way to divide a surface into regions of equal area with the least total perimeter. Hales announced the feat last month and posted his proof on the Internet at http://www.math.lsa.umich.edu/~hales/.

Although widely believed and often asserted as fact, this conjecture has long eluded proof, says Frank Morgan of Williams College in Williamstown, Mass. Hales' proof "looks right to me," he comments, "although I have not checked every detail."

Last year, Hales proved Johannes Kepler's conjecture that the arrangement of the familiar piles of neatly stacked oranges at a supermarket represents the best way to pack identical spheres tightly (SN: 8/15/98, p. 103).

If Hales' proofs of the honeycomb and Kepler conjectures stand the test of time, "it's a remarkable double achievement," says physicist Denis Weaire of Trinity College Dublin in Ireland.

In an essay on the "sagacity of bees," Pappus of Alexandria noted in the fourth century A.D. how bees, possessing a divine sense of symmetry, had as their mission the fashioning of honeycombs without any cracks through which that wonderful nectar known as honey could be lost. In his mathematical analysis, he focused on the hexagonal arrangement of cells.

Although honeycomb cells are three-dimensional structures, each cell is uniform in the direction perpendicular to its base. Hence, its hexagonal cross section matters more than other factors in calculating how much wax it takes to construct a comb.

The mathematicians' honeycomb conjecture therefore concerns a two-dimensional pattern—as if bees were creating a grid for laying out tiles to cover an infinitely wide bathroom floor.

Mathematicians of ancient Greece asked what choices bees might have if they wanted to divide a flat surface into identical, equal-sided cells. Only three regular polygons pack together snugly without leaving gaps: equilateral triangles, squares, and regular hexagons.

Other polygons, such as pentagons and octagons, will not fit together without leaving spaces between the cells.

The Greeks asserted that if the same quantity of wax were used for the construction of a single three-dimensional version of the three candidate figures, the hexagonal cell would hold more honey than a triangular or square cell. Equivalently, the perimeter of a hexagonal cell enclosing a given area is less than that of a square or triangular cell enclosing the same area.

Other possibilities for arrays of cells, however, are conceivable. There's no a priori reason why the cells must all have equal sides or identical shapes and sizes. What about a crazy quilt of random polygons or cells with curved rather than straight sides?

Sorting through these alternative patterns proved a formidable task for mathematicians.

It was relatively straightforward to establish that a regular hexagon, with equal sides and 120° angles, has a smaller perimeter than any other six-sided figure of the same area. Moreover, polygons with more sides than the hexagon, such as regular octagons, do better, and polygons with fewer sides, such as squares, do worse.

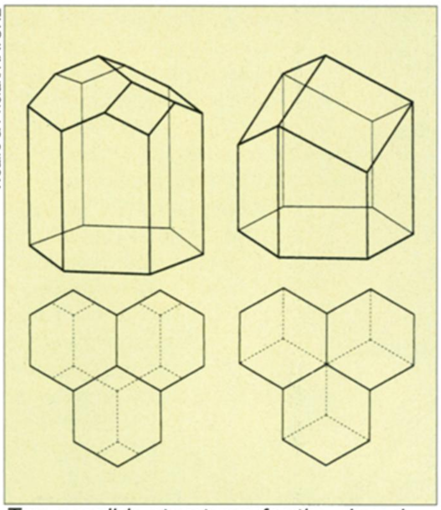
In 1943, Hungarian mathematician L. Fejes Tóth proved the honeycomb conjecture for the special case of filling the plane with any mixture of straight-sided polygons. In effect, Morgan says, Tóth established that the average number of sides per cell in a plane-filling pattern is at most six. Moreover, the advantage of having some polygons with more than six sides is less than the disadvantage of having some polygons with fewer sides. Under these conditions, the least-perimeter way to enclose and separate infinitely many regions of equal area is the regular hexagonal grid of the honeycomb.

What if cells were allowed to have curved sides? Tóth considered the question and predicted that the best answer is still a grid of regular hexagons. "Nevertheless, this conjecture has resisted all attempts at proving it," he commented.

In recent years, Morgan has refocused attention on the honeycomb conjecture and related questions, such as the most economical way of packaging a pair of identical volumes as double bubbles (SN: 8/12/95, p. 101). In the May Transactions of the American Mathematical Society, he outlined progress in proving the hexagonal honeycomb conjecture and its variants, and he suggested a possible route to a proof.

With curved sides, the complication is that a side that bulges out for one cell must bulge in for its immediate neighbor. Bulging out helps minimize the cell perimeter, while bulging in hurts.

Hales proved that the advantage of bulging out is less than the disadvantage of bulging in. "The basic idea is quite simple and elegant," says John M. Sullivan of the University of Illinois at Urbana-Champaign. Hales' main result "shows that no single cell can do better than a hexagon if appropriately penalized for having more than six sides or outward curves."



Weaire & Phelan/NATURE

Two possible structures for the closed end of a honeycomb cell. Mathematician L. Fejes Tóth showed that an end cap consisting of two hexagons and two squares (top left) requires a little less wax than the one honeybees make, with three diamond-shaped, or rhombic, panels (top right). A honeycomb consists of two layers of such cells placed back to back so that a chamber on one side is offset from its partner on the other side (bottom).

Therefore, straight-sided polygons work better than curved ones, and regular hexagons are truly best of all.

Bees have that aspect of their honeycomb structures down pat.

There's more to a honeycomb than a vertical, hexagonal grid, however. It actually consists of two layers of cells placed back to back. The cells themselves are tilted upward at an angle of about 13° from the horizontal—just enough to prevent stored honey from dripping out.

Instead of a flat bottom, each cell ends in three four-sided, diamond-shaped panels, meeting in a point like a pencil sharpened with only three knife strokes. The cells of the two layers are offset so the center of a chamber on one side is the corner of three adjacent cells on the other side. This allows the layers to interlock like the bottoms of two egg cartons fitted together. In the honeycomb, however, one layer of material serves as the bottoms of two cells. In cross section, the interface between the two layers has a zigzag structure.

The angles of each diamond-shaped, or rhombic, face of the cell bottom are 109.5° and 70.5°. In the 18th century, mathematicians proved that these particular angles give the maximum volume for a three-rhombus configuration.

In 1964, Tóth discovered that a combination of two hexagons and two squares does a little better than an end cap of three rhombuses in terms of the efficient use of wax. The difference, however, is very small. "By building such cells, the bees would save per cell less than 0.35 percent of the area of an opening (and a much smaller percentage of the surface area of a cell)," he concluded.

Several years ago, Weaire and his colleague Robert Phelan experimented with a liquid-air foam to test Tóth's mathematical model. They pumped equal-sized bubbles, about 2 mm in diameter, of a detergent solution between two glass plates to generate a double layer.

The two layers of trapped bubbles formed hexagonal patterns at the glass plates. The interface between the two layers adopted Tóth's structure.

When Weaire and Phelan thickened the bubble walls by adding more liquid, however, they unexpectedly found an abrupt transition. When the walls reached a particular thickness, the interface suddenly switched to the three-rhombus configuration of a honeycomb.

The switch also occurs in the reverse direction as liquid is removed.

So, honeybees may very well have found the optimal design solution for the thicker wax walls of their honeycomb cells.

For mathematicians, however, "many questions remain open," Morgan says.

In two dimensions, for example, mathematicians can consider what happens when they allow arrangements that include regions of several, intermingled components or empty spaces between cells. In three dimensions, the question of what space-filling arrangement of cells of equal size has the minimum surface area is still not settled (SN: 3/5/94, p. 149).

"The strategies I developed for the Kepler conjecture were very useful with the honeycomb conjecture," Hales says. "A topic for future research might be to determine to what extent [those methods] can be adapted to other optimization problems."

These are matters that concern not only mathematicians but also researchers interested in the characteristics and behavior of fluids, bubbles, foams, crystals, and a variety of biological structures, from cell assemblages to plant tissue.

"Cell and tissue, shell and bone, leaf and flower, are so many portions of matter, and it is obedience to the laws of physics that their particles have been moved, moulded and conformed," D'Arcy W. Thompson wrote in his celebrated book *On Growth and Form*, first published in 1917. "Their problems of form are in the first instance mathematical problems, their problems of growth are essentially physical problems."

The honeybee's honeycomb fits neatly into the atlas of mathematically optimal forms found in nature. □