# Portraits of Equations

By IVARS PETERSON

*"The mathematician's patterns, like the painter's or the poet's, must be beautiful...."*
— G.H. Hardy

For many mathematicians, the words "beauty" and "mathematics" go side by side. These mathematicians strive not just to construct irrefutable proofs but also to present their ideas and results in a clear and compelling fashion — a fashion dictated more by a sense of aesthetics than by the needs of logic.

Computer graphics has added a new element to the beauty of mathematics. Over the last few years, mathematicians have begun to explore and enjoy the patterns, made visible by computer graphics, in their equations and other mathematical formulations. Using computer-based techniques, they have discovered graceful geometric forms reminiscent of soap-film surfaces (SN: 3/16/85, p.168), studied the bizarre, chaotic results of iterating simple equations (SN: 2/28/87, p.137), visualized higher dimensions and penetrated the infinitely detailed world of fractals (SN: 5/2/87, p.283).

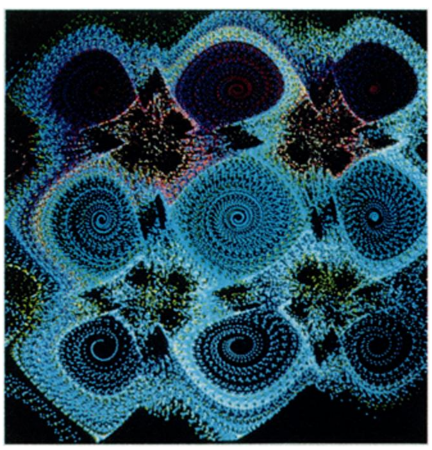
Computer graphics also allows nonmathematicians to experience a little of the pleasure that mathematicians take in their work. While mathematicians use such images to inspire and further their research, nonmathematicians are able to appreciate some of the mathematical qualities portrayed in the pictures. In fact, anyone with a little imagination, some programming skill and access to a computer can generate breathtaking images of mathematical objects.

One major source of striking images is the mathematical exploration of dynamical systems. This involves tracking the behavior of mathematical expressions called differential equations, which describe the way mathematically characterized systems change over time. In a sense, a set of differential equations is like a machine that takes in values for all the variables and then generates the new values at some later time. When such equations are used to represent physical systems, such as the flow of water past an obstacle, the variables may stand for characteristics such as position and velocity (SN: 7/30/83, p.76).

Often, the relationship expressed in the equations is nonlinear; that is, input and output are not proportional. Mathematicians have learned that, under the right conditions, even simple sets of nonlinear differential equations can yield numbers that appear to follow no pattern. Although the equations express direct cause-and-effect relationships, the numerical results predict that modeled systems can show irregular motion or random-like, chaotic behavior.

In fact, this class of solutions displays a sensitive dependence on initial conditions. A slightly different starting point produces a radically different result. In principle, the future is completely determined by the past, but in practice, small uncertainties are amplified, so that even though the behavior is predictable in the short term, it is unpredictable in the long term.

This behavior can best be seen in phase space, where each dimension represents one of the variables in the differential equations. If there are only two variables, *x* and *y*, successive points can be plotted using a simple coordinate system to locate each point on a flat piece of paper or a computer screen.



Pickover/IBM

*Pickover works with a set of differential equations in two variables (x and y), which can be represented as a pair of difference equations: $x_{t+1} = x_t - hf(y_t)$ and $y_{t+1} = y_t + hf(x_t)$. In other words, with time steps of size h, the next or $t+1$ value of the variable x is equal to the old value at time t minus the value of a certain mathematical expression or function when y has a particular value. In the second equation, the variable y is treated in roughly the same way.*

*For his function, Pickover uses expressions of the form $f(x) = \sin[x + \sin(kx)]$. The sine function typically looks like a smoothly curving set of waves. Here, one set of waves is modulated by another set of waves with a different frequency (governed by k). The top picture is a magnification of part of the portrait for $f(x) = \sin[x + \tan(3x)]$. The lower image shows the portrait derived from a slightly different set of difference equations and the function $f(x) = \sin[x + \sin(3x + \sin(2x))]$.*



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Beginning at a point representing the initial values of all the variables, the differential equation generates a trajectory that winds through its particular phase space. The location of a point on the trajectory at any time contains all the information needed to describe the system's state (that is, the values of all the variables) at that particular time. Researchers are interested in what happens to these phase-space trajectories for different equations under various circumstances.

Computer scientist Clifford A. Pickover of the IBM Thomas J. Watson Research Center In Yorktown Heights, N.Y., is one of a growing number of people interested in exploring the visually appealing side of mathematics. Pickover's interest in generating beautiful mathematical patterns complements his efforts to develop improved methods of representing complex data (SN: 6/20/87, p.392). Pickover is also editor of THE JOURNAL OF CHAOS AND GRAPHICS, an informal publication that looks at how complicated behavior and structures can arise in systems based on simple rules.

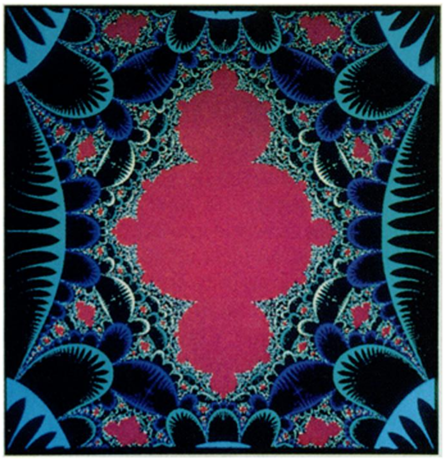
One of Pickover's more recent efforts has been to study the behavior of a special set of differential equations, using mathematical expressions that model modulated radio waves. His work was inspired by earlier research done by Roger D. Nussbaum of Rutgers University in New Brunswick, N.J., and Heinz-Otto Peitgen of the University of California at Santa Cruz, who examined the same set of differential equations using other mathematical expressions. Pickover's results appear in the current COMPUTERS & GRAPHICS (Vol.11, No.2).

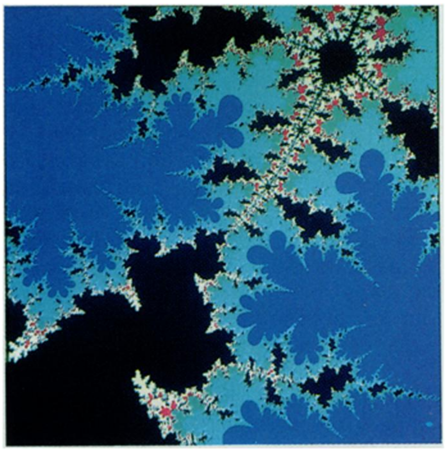
To see what happens, Pickover generates "phase portraits" of the equations. He selects a starting point, plots it, then computes its coordinates a short time later, plots the new value, and continues the procedure for as many steps as he chooses. The result is a sweeping line of dots across his computer screen. Each different starting point generates a new line. Eventually, the screen is covered with swirls of color (see cover).

Pickover's innovation has been to make the procedure of plotting the behavior of this type of differential equation rapid and interactive. He can easily set the picture boundaries, the number of iterations or time steps, the size of the time steps and the picture resolution. He uses color to distinguish different trajectories.

"The interactive system," says Pickover, "allows the user to choose parameters best suited for visual demonstrations of features of interest."

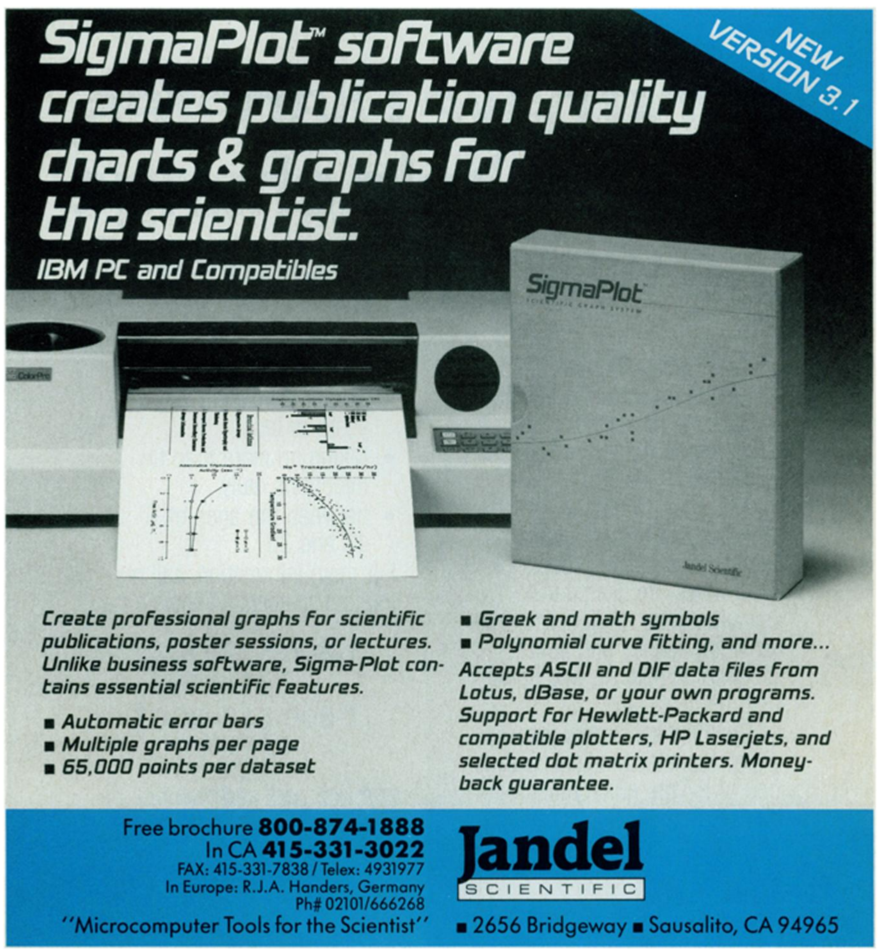
Another aspect of mathematics worth investigating visually is the behavior of mathematical expressions known as transcendental functions (SN: 5/26/84, p.328). Members of the





*The transcendental function known as the hyperbolic cosine (cosh) can be explored in two ways. One is to see what the Julia set is for a particular expression cosh(z) + c, where c is some constant chosen arbitrarily and both c and z are complex numbers. The expression cosh(z) can also be written as* $\frac{e^z + e^{-z}}{2}$. *The top picture shows the Julia set (colored area) when c = (-2.25,0).*

*Another approach is to see what happens to the Julia sets for various values of c. Plotting in black the values of c for which the sequence of complex numbers starting at the point (0,0) fails to escape to infinity produces a new, extremely intricate picture. The bottom illustration shows a magnification of one portion of the c-plane picture. The varying colors indicate the rate at which points outside the black area escape to infinity.*

family of transcendental functions include the exponential, sine and cosine functions. The exponential function, represented by $e$ to some power $x$, is familiar to anyone who has dealt with compounded growth, whether in populations or in accumulated interest in a savings account at a bank. The technique for studying the behavior of these functions is equivalent to entering a number into the display of a scientific calculator, locating the appropriate key, then repeatedly pressing that button, all the time observing what happens to the successive numbers displayed.

The most interesting results are seen when the investigator deals with complex numbers rather than with the ordinary real numbers that a calculator uses. Complex numbers make it possible to wander across a broad plane rather than along a narrow road. Each iteration represents a step along a path that hops from one complex number, $z$, to the next. The collection of all such points along a path constitutes an orbit. The basic goal is to understand the ultimate fate of all orbits for a given system.

Depending on the value of $z$ chosen as a starting point, the orbit may behave in one of several different ways. It may rapidly converge to a single point and stay there; in other words, the same number comes up again and again. Alternatively, it may always return to a certain value after a fixed number of iterations. Or the numbers may get steadily larger.

In fact, the starting points of orbits can be color-coded to indicate how quickly the points escape along their orbits to infinity. In contrast, points that tend to stay close to their starting values are usually shown in black. The colored area for the iteration of a particular mathematical expression is known as a Julia set. These sets often look spectacular.

Pickover's contribution to the study of the dynamical behavior of transcendental functions was examination of the behavior of the hyberbolic cosine (cosh) function in the complex plane. His graphics experiments, he says, "are good ways to show the complexity of the transition region between convergence and divergence."

"The process of iteration," says Pickover, "can be likened to pulling layers from a fruit whose center contains a hard kernel." That kernel is what's left after an infinite number of iterations and has an extremely convoluted and complex boundary. As pictured on a computer screen, points that fall within black regions (that is, within the kernel) have different fates upon iteration than those on the outside.

Says Pickover, "Computers with graphics have played a critical role in the study of iterated sets and in helping mathematicians form the intuitions needed to prove new theorems about convergence of points in the complex plane." □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